THE PRIME NUMBER THEOREM

PO-LAM YUNG

1. Introduction

Let $\pi(x)$ be the number of primes $\leq x$. The famous prime number theorem, first proved by Hadamard and de la Vallèe Poussin, asserts the following:

Theorem 1 (Prime number theorem)**.**

$$\pi(x) \sim \frac{x}{\log x} \tag{1}$$

as $x \to +\infty$. *(This means* $\lim_{x\to+\infty}(\pi(x)\log x)/x = 1$*).*

In Chapter 7 of [4], a complex analytic proof of the prime number theorem was given, based on analysis of the Chebychev's ψ function:

$$\psi(x) = \sum_p \sum_{m\in\mathbb{N}\colon p^m\leq x} \log p = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p,$$

where $[x]$ is the greatest integer less than or equal to x. (Any sum or product over p in this article is a sum or product over all primes p.) There they used some estimates of ζ near the line $\operatorname{Re} s = 1$ (more precisely, an upper bound for $|\zeta'(s)/\zeta(s)|$). On the other hand, in [2], Newman gave another complex analytic proof of the prime number theorem, using only the vanishing of ζ on the line $\operatorname{Re} s = 1$, by considering the φ function:

$$\varphi(x) = \sum_{p\leq x} \log p.$$

(See also the exposition in [5].) In this article, we try to combine the two approaches, and adapt Newman's argument so that it works through Chebychev's ψ function (rather than the φ function). We note in passing that elementary approaches to the prime number theorem are also possible, most notably by Erdös [1] and Selberg [3]. But we will not discuss these here.

Recall that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{valid for } \operatorname{Re} s > 1.$$

We will assume known that ζ has a meromorphic continuation to $\mathbb{C}$, so that

(i) the only singularity of ζ in $\mathbb{C}$ is a simple pole at $s = 1$, and
(ii) the residue of ζ at $s = 1$ is 1.

Date: November 5, 2015.

In other words, $\zeta(s) - \frac{1}{s-1}$ is entire. This can be done, for instance, by establishing the functional equation of $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. We also assume known the following product factorization of ζ over all primes:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}, \quad \text{valid for } \operatorname{Re} s > 1.$$

2. Non-vanishing of ζ on $\operatorname{Re} s = 1$

We now prove the following theorem.

Theorem 2. *ζ has no zeroes on the vertical line where $\operatorname{Re} s = 1$.*

Proof. We need three observations.

First, note that ζ is real-valued on $\{s \in \mathbb{R} \colon s > 1\}$. Hence

$$\overline{\zeta(s)} = \zeta(\overline{s}) \quad \text{for all } s \in \mathbb{C}.$$

This shows that the (non-real) zeroes of ζ comes in conjugate pairs: if s is a zero of ζ, then so is $\overline{s}$, and ζ vanishes to the same order at both s and $\overline{s}$.

Next, recall that if f is a meromorphic function near a point z_0, then the order of of f at z_0 (which is positive if f vanishes there, negative if f has a pole there) can be computed via the residue of the logarithmic derivative of f at z_0. In particular, for any $t \in \mathbb{R}$, the order of ζ at $1+it$ is

$$\operatorname{ord}_{1+it}\zeta = \lim_{\epsilon \to 0} \frac{\epsilon\, \zeta'(1+it+\epsilon)}{\zeta(1+it+\epsilon)}. \tag{2}$$

Finally, we note that the product factorization of ζ allows one to compute the logarithmic derivative of ζ:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{p^{-s}\log p}{1-p^{-s}} = -\sum_p \sum_m \frac{\log p}{p^{ms}}, \quad \text{valid for } \operatorname{Re} s > 1.$$

where the sum over m is over all positive integers m. In particular, introducing the von Mangoldt function Λ, where

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and some positive integer } m \\ 0, & \text{otherwise} \end{cases},$$

we see that the logarithmic derivative of ζ is a Dirichlet series, whose coefficients are precisely given by Λ:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{valid for } \operatorname{Re} s > 1. \tag{3}$$

What will be important for us is that $\Lambda(n)$ is real and non-negative for all positive integers n.

Now we are ready to put all these together. First, since ζ has a pole at $s = 1$, it cannot have a zero at $s = 1$. Next, suppose ζ is zero at $s = 1 + it$ for some $t \in \mathbb{R}$. We show that this is impossible by considering the orders of ζ at $1 \pm 2it$, $1 \pm it$ and 1: Combining (2) and (3), we see that

$$\begin{aligned}
\mathrm{ord}_{1+2it}\zeta &= -\lim_{\epsilon\to 0^+} \epsilon \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} n^{-2it} \\
\mathrm{ord}_{1+it}\zeta &= -\lim_{\epsilon\to 0^+} \epsilon \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} n^{-it} \\
\mathrm{ord}_{1}\zeta &= -\lim_{\epsilon\to 0^+} \epsilon \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} \\
\mathrm{ord}_{1-it}\zeta &= -\lim_{\epsilon\to 0^+} \epsilon \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} n^{it} \\
\mathrm{ord}_{1-2it}\zeta &= -\lim_{\epsilon\to 0^+} \epsilon \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} n^{2it}
\end{aligned}$$

We now multiply these five equations by 1, 4, 6, 4, 1 respectively, and add them all up. Observe that

$$n^{-2it} + 4n^{-it} + 6 + 4n^{it} + n^{2it} = (n^{it/2} + n^{-it/2})^4 = (2\cos(t \log n/2))^4 \geq 0. \tag{4}$$

Since $\Lambda(n) \geq 0$ for all n, we then see that

$$\mathrm{ord}_{1+2it}\zeta + 4\,\mathrm{ord}_{1+it}\zeta + 6\,\mathrm{ord}_{1}\zeta + 4\,\mathrm{ord}_{1-it}\zeta + \mathrm{ord}_{1-2it}\zeta \leq 0.$$

But

$$\mathrm{ord}_{1}\zeta = -1,$$
$$\mathrm{ord}_{1+it}\zeta = \mathrm{ord}_{1-it}\zeta,$$

and

$$\mathrm{ord}_{1+2it}\zeta = \mathrm{ord}_{1-2it}\zeta \geq 0.$$

Hence

$$8\,\mathrm{ord}_{1+it}\zeta - 6 \leq 0,$$

which contradicts our assumption that $\zeta(1 + it) = 0$. □

We remark that (4) is really Lemma 1.4 of Chapter 7 of [4] in disguise. Also, by rewriting ζ'/ζ as the derivative of $\log \zeta$, and undoing the derivative, the above argument essentially gives Corollary 1.5 of Chapter 7 of [4].

3. Proof of the prime number theorem

We are now ready for the proof of the prime number theorem. First, we recall Chebyshev's ψ function:

$$\psi(x) = \sum_{p} \sum_{m\in\mathbb{N}\colon p^m \leq x} \log p = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p.$$

It is well known that the prime number theorem is equivalent to the assertion that

$$\psi(x) \sim x. \tag{5}$$

Indeed, assume for the moment that (5) holds. Then since

$$\psi(x) \leq \sum_{p\leq x} \log x = \pi(x) \log x,$$

dividing both sides by x and letting $x \to +\infty$, we get

$$1 \leq \liminf_{x\to\infty} \frac{\pi(x)\log(x)}{x}.$$

Also, for any $\alpha \in (0,1)$, we have

$$\psi(x) \geq \sum_{p\leq x} \log p \geq \sum_{x^\alpha<p\leq x} \log p \geq (\pi(x)-\pi(x^\alpha))\log(x^\alpha) \geq \alpha(\pi(x)-x^\alpha)\log x.$$

Hence if (5) holds, then dividing the above inequality by x, and letting $x \to \infty$, we get that

$$1 \geq \alpha \limsup_{x\to+\infty} \frac{(\pi(x)-x^\alpha)\log x}{x} = \alpha \limsup_{x\to+\infty} \frac{\pi(x)\log x}{x}.$$

Letting $\alpha \to 1^-$, we get

$$1 \geq \limsup_{x\to+\infty} \frac{\pi(x)\log x}{x}.$$

Together we obtain (1), and the prime number theorem holds.

The converse implication, namely that (1) implies (5), is not much harder. Since we do not need this direction of the implication, we leave this verification to the interested reader.

Now to prove the prime number theorem, it suffices to verify (5). In order to show that $\psi(x) \sim x$ as $x \to +\infty$, it may help to first verify a weaker statement, namely that $\psi(x)/x$ remains bounded as $x \to +\infty$. This is what we are going to do next.

Proposition 3. $\frac{\psi(x)}{x}$ *remains bounded as* $x \to +\infty$.

Proof. First, we claim that there exists a constant C, such that for any positive integer n, we have

$$\psi(2n) - \psi(n) \leq Cn. \tag{6}$$

To prove this claim, note that

$$\psi(2n)-\psi(n) = \sum_p \sum_{\{m\colon n<p^m\leq 2n\}} \log p = \log\left(\prod_{m=1}^{\infty} \prod_{\{p\colon n<p^m\leq 2n\}} p\right). \tag{7}$$

In the product inside the logarithm, consider first the term corresponding to $m = 1$. We have

$$\prod_{\{p\colon n<p\leq 2n\}} p \leq \binom{2n}{n}. \tag{8}$$

Indeed

$$\binom{2n}{n} = \frac{(2n)(2n-1)\dots(n+1)}{n!}$$

is an integer, so that $n!$ is a factor of $(2n)(2n-1)\dots(n+1)$; also each prime p with $n < p \leq 2n$ is a factor of $(2n)(2n-1)\dots(n+1)$. Since each such prime p is relatively prime with $n!$, we see that $(n!)\prod_{\{p\colon n<p\leq 2n\}} p$ divides $(2n)(2n-1)\dots(n+1)$, i.e. $\prod_{\{p\colon n<p\leq 2n\}} p$ divides $\binom{2n}{n}$. In particular, (8) holds. This further implies

$$\prod_{\{p\colon n<p\leq 2n\}} p \leq 2^{2n}, \tag{9}$$

since

$$\binom{2n}{n} \le \sum_{k=0}^{2n} \binom{2n}{k} = (1+1)^{2n} = 2^{2n}.$$

This completes our estimate of the product inside the logarithm on the right hand side of (7).

Next, we consider those terms in the same product corresponding to $m \ge 2$. If $m \ge 2$, and $n < p^m \le 2n$, then $p \le \sqrt{2n}$. Also, for each prime p, there is at most one power of p that lies in $(n, 2n]$. Hence

$$\prod_{m=2}^{\infty} \prod_{\{p \colon n < p^m \le 2n\}} p \le \prod_{p \le \sqrt{2n}} p \le (\sqrt{2n})^{\sqrt{2n}}.$$

Hence, together with (9), we obtain

$$\psi(2n) - \psi(n) \le \log\left(2^{2n}(\sqrt{2n})^{\sqrt{2n}}\right) \le 2n \log 2 + \frac{\sqrt{2n}\log(2n)}{2}.$$

This establishes our claim (6).

Now that we have the claim (6), we see that there exists a constant C' such that

$$\psi(2x) - \psi(x) \le C'x$$

for all $x \ge 1$. In fact it suffices to prove this for x large. To do so, take n to be the integer closest to x. Then the sum defining $\psi(x)$ and $\psi(n)$ differ in at most one term, and $|\psi(x) - \psi(n)| \le C \log x \le C''x$. Similarly, $|\psi(2x) - \psi(2n)| \le C''x$. Hence together with the bound for $\psi(2n) - \psi(n)$ we already established, we see that $\psi(2x) - \psi(x) \le C'x$, as desired.

Now we just iterate this estimate:

$$\begin{aligned} \psi(x) - \psi(x/2) &\le C'(x/2) \\ \psi(x/2) - \psi(x/4) &\le C'(x/4) \\ &\vdots \end{aligned}$$

and sum up a geometric series on the right. Then

$$\psi(x) \le C'x$$

as $x \to +\infty$, as desired. $\square$

The function $\psi(x)$, defined for $x > 1$, can be identified with a function $\psi(e^t)$, defined for $t > 0$. The Laplace transform of the latter is then given by

$$\int_0^\infty \psi(e^t)e^{-st}dt = \int_1^\infty \frac{\psi(x)}{x}\frac{dx}{x^s}.$$

It turns out that the latter can be expressed in terms of the ζ function. More precisely, we have:

Proposition 4.

$$\int_1^\infty \frac{\psi(x)}{x}\frac{dx}{x^s} = -\frac{\zeta'(s)}{s\zeta(s)}, \quad \textit{valid for } Re\, s > 1.$$

Proof. Just note that when $\mathrm{Re}\, s > 1$, we have

$$\int_1^\infty \frac{\psi(x)}{x}\frac{dx}{x^s} = \int_1^\infty \sum_p \sum_{p^m \le x} \log p \frac{dx}{x^{s+1}} = \sum_p \sum_m \int_{p^m}^\infty \log p \frac{dx}{x^{s+1}} = \frac{1}{s}\sum_p \sum_m \frac{\log p}{p^{ms}} = -\frac{\zeta'(s)}{s\zeta(s)}.$$

□

We are interested in showing $\dfrac{\psi(x)}{x} - 1 \to 0$ as $x \to +\infty$. Hence we are led to consider the following identity:

Proposition 5.

$$\int_1^\infty \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x^s} = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}, \quad \textit{valid for } \mathrm{Re}\, s > 1.$$

Proof. This follows from Proposition 4 by simply noting that

$$\int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1}, \quad \text{for } \mathrm{Re}\, s > 1.$$

□

We note that the right hand side of the identity in Proposition 5 actually extends to a holomorphic function on an open set containing the closed half plane $\{\mathrm{Re}\, s \geq 1\}$. Thus the following principle applies:

Proposition 6. *Let $f(x)$ be a bounded function on $[1,\infty)$, and define*

$$g(s) = \int_1^\infty f(x)\frac{dx}{x^s} \quad \textit{for } \mathrm{Re}\, s > 1.$$

Then g is holomorphic on $\mathrm{Re}\, s > 1$. If g extends to a holomorphic function on an open set containing the closed half plane $\mathrm{Re}\, s \geq 1$, then $\int_1^\infty f(x)\frac{dx}{x}$ exists, and is equal to $g(1)$.

Indeed

$$f(x) := \frac{\psi(x)}{x} - 1$$

is a bounded function by Proposition 3, and the integral

$$\int_1^\infty f(x)\frac{dx}{x^s} = \int_1^\infty \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x^s}$$

extends to a holomorphic function on an open set containing the closed half plane $\{\mathrm{Re}\, s \geq 1\}$ by Proposition 5. Hence assuming Proposition 6 for the moment, we obtain the following proposition:

Proposition 7. *The improper integral* $\displaystyle\int_1^\infty \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x}$ *converges.*

This allows us to finish the proof of the prime number theorem.

Proof of Theorem 1. As observed before, it suffices to verify (5). We argue by contradiction. Suppose (5) is false. Then either there exists $\alpha > 1$ such that $\psi(x_n) > \alpha x_n$ for a sequence $\{x_n\}$ with $x_n \to +\infty$, or there exists $\beta < 1$ such that $\psi(y_n) < \beta y_n$ for a sequence $\{y_n\}$ with $y_n \to +\infty$. In the first case, since ψ is an increasing function, we have $\psi(x) \geq \psi(x_n) \geq \alpha x_n$ whenever $x \geq x_n$. In particular,

$$(10) \qquad \int_{x_n}^{\alpha x_n} \left(\frac{\psi(x)}{x} - 1\right)\frac{dx}{x} \geq \int_{x_n}^{\alpha x_n} \left(\frac{\alpha x_n}{x} - 1\right)\frac{dx}{x} = \int_1^\alpha \left(\frac{\alpha}{x} - 1\right)\frac{dx}{x},$$

the last integral being strictly positive, and independent of n. This contradicts Proposition 7: in fact, Proposition 7 implies that

$$\int_{x_n}^{\alpha x_n} \left(\frac{\psi(x)}{x} - 1 \right) \frac{dx}{x} = \int_1^{\alpha x_n} \left(\frac{\psi(x)}{x} - 1 \right) \frac{dx}{x} - \int_1^{x_n} \left(\frac{\psi(x)}{x} - 1 \right) \frac{dx}{x} \to 0$$

as $n \to \infty$, and this is not compatible with the lower bound we have obtained in (10).

Similarly, in the second case, we use $\psi(x) \leq \psi(y_n) < \beta y_n$ whenever $x \leq y_n$, to conclude that

$$\int_{\beta y_n}^{y_n} \left(\frac{\psi(x)}{x} - 1 \right) \frac{dx}{x} \leq \int_{\beta y_n}^{y_n} \left(\frac{\beta y_n}{x} - 1 \right) \frac{dx}{x} = \int_{\beta}^{1} \left(\frac{\beta}{x} - 1 \right) \frac{dx}{x} < 0$$

independent of n. This contradicts Proposition 7. $\square$

It remains to prove Proposition 6.

Proof of Proposition 6. Suppose f is bounded, say $|f(x)| \leq M$ for all $x \geq 1$. Suppose also that

$$g(s) := \int_1^\infty f(x) \frac{dx}{x^s}$$

extends holomorphically to an open set containing the closed half plane $\{\operatorname{Re} s \geq 1\}$. Let

$$g_t(s) = \int_1^t f(x) \frac{dx}{x^s}.$$

Then g_t is entire for all t, and our goal is to show that $g_t(1)$ converges to $g(1)$ as $t \to +\infty$. For $\varepsilon > 0$ and $\delta > 0$, let $\Lambda_{\varepsilon,\delta}$ be the positively oriented closed contour, given by

$$\Lambda_{\varepsilon,\delta} = C_\varepsilon + L_\delta^{(1)} + L_\delta^{(2)} + L_\delta^{(3)} \tag{11}$$

where

- C_ε be the semicircle in the right half plane $\{\operatorname{Re} s > 1\}$, that is centered at 1 and of radius $1/\varepsilon$;
- $L_\delta^{(1)}$ is the horizontal straight line joining $1 + i\varepsilon^{-1}$ to $1 - \delta + i\varepsilon^{-1}$;
- $L_\delta^{(2)}$ is the vertical straight line joining $1 - \delta + i\varepsilon^{-1}$ to $1 - \delta - i\varepsilon^{-1}$; and
- $L_\delta^{(3)}$ is the horizontal straight line joining $1 - \delta - i\varepsilon^{-1}$ to $1 - i\varepsilon^{-1}$.

Then for any $\varepsilon > 0$, as long as δ is sufficiently small, we have, by Cauchy integral formula, that

$$g_t(1) - g(1) = \frac{1}{2\pi i} \int_{\Lambda_{\varepsilon,\delta}} [g_t(s) - g(s)] \frac{ds}{s-1}.$$

For various technical reasons, we will actually use the following identity instead (which also follow from Cauchy's integral formula, since the extra factor $t^{s-1}(1 + \varepsilon^2(s-1)^2)$ is entire in s, and equals 1 when $s = 1$):

$$g_t(1) - g(1) = \frac{1}{2\pi i} \int_{\Lambda_{\varepsilon,\delta}} [g_t(s) - g(s)] t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} \tag{12}$$

Now we decompose the above path integral into 4 parts, according to (11). For $s \in C_\varepsilon$, we have

$$\begin{aligned} |g_t(s) - g(s)| &\le \int_t^\infty |f(x)| \frac{dx}{x^{\mathrm{Re}\, s}} \le \frac{M t^{1-\mathrm{Re}\, s}}{\mathrm{Re}\, s - 1}, \\ |t^{s-1}| &\le t^{\mathrm{Re}\, s - 1} \\ |1 + \varepsilon^2 (s-1)^2| &= \frac{|s - (1 - i\varepsilon^{-1})||s - (1 + i\varepsilon^{-1})|}{\varepsilon^{-2}} \le C \frac{|\mathrm{Re}\, s - 1|}{\varepsilon^{-1}} \\ \frac{1}{|s-1|} &= \frac{1}{\varepsilon^{-1}}. \end{aligned}$$

Hence

$$\left| \frac{1}{2\pi i} \int_{C_\varepsilon} [g_t(s) - g(s)] t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} \right| \le CM\varepsilon \tag{13}$$

Next, let $\tilde{C}_\varepsilon$ be the semi-circle in the left half plane $\{\mathrm{Re}\, s < 1\}$, that is centered at 1 and of radius $1/\varepsilon$. Then we integrate the part concerning g in (12), over $\tilde{C}_\varepsilon$ instead of over $L_\delta^{(1)} + L_\delta^{(2)} + L_\delta^{(3)}$. (This is possible because g_t is entire.) Hence

$$\int_{L_\delta^{(1)} + L_\delta^{(2)} + L_\delta^{(3)}} g_t(s) t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} = \int_{\tilde{C}_\varepsilon} g_t(s) t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1}.$$

But on $\tilde{C}_\varepsilon$, we have

$$|g_t(s)| \le \int_1^t |f(x)| \frac{dx}{x^{\mathrm{Re}\, s}} \le \frac{M t^{1-\mathrm{Re}\, s}}{1 - \mathrm{Re}\, s}.$$

Similarly as before, on $\tilde{C}_\varepsilon$, we have

$$|t^{s-1}(1 + \varepsilon^2 (s-1)^2)| \le C \frac{t^{\mathrm{Re}\, s - 1}(1 - \mathrm{Re}\, s)}{\varepsilon^{-1}}.$$

Hence

$$\left| \frac{1}{2\pi i} \int_{L_\delta^{(1)} + L_\delta^{(2)} + L_\delta^{(3)}} g_t(s) t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} \right| \le CM\varepsilon. \tag{14}$$

Finally, the contribution of $g(s)$ to the contour integral over $L_\delta^{(1)} + L_\delta^{(3)}$ is given by

$$\left| \frac{1}{2\pi i} \int_{L_\delta^{(1)} + L_\delta^{(3)}} g(s) t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} \right| \le C\delta\varepsilon \tag{15}$$

where $|g| \le C$ on $L_\delta^{(1)} \cup L_\delta^{(3)}$. This is because $L_\delta^{(1)}$ and $L_\delta^{(3)}$ both have lengths $\le \delta$, and that $1/|s-1| \simeq \varepsilon$ on $L_\delta^{(1)} \cup L_\delta^{(3)}$. (Note also that $|t^{s-1}| \le 1$ since $\mathrm{Re}\, s < 1$, and $|1 + \varepsilon^2 (s-1)^2| \le C$ on $L_\delta^{(1)} \cup L_\delta^{(3)}$.) Now the contribution of $g(s)$ to the contour integral over $L_\delta^{(2)}$ is given by

$$\left| \frac{1}{2\pi i} \int_{L_\delta^{(2)}} g(s) t^{s-1} (1 + \varepsilon^2 (s-1)^2) \frac{ds}{s-1} \right| \le C \frac{t^{-\delta}}{\varepsilon\delta}. \tag{16}$$

This is because the length of $L_\delta^{(2)}$ is $2/\varepsilon$, the function $|g|$ is bounded by C on $L_\delta^{(2)}$, and $|t^{s-1}| = t^{-\delta}$ on $L_\delta^{(2)}$; also, $|1 + \varepsilon^2 (s-1)^2| \le C$ on $L_\delta^{(2)}$, and $1/|s-1| \le 1/\delta$ on $L_\delta^{(2)}$.

Altogether, by (12), (13), (14), (15) and (16), we see that for any $\varepsilon > 0$, there exists a small $\delta > 0$, such that

$$|g_t(1) - g(1)| \leq CM\varepsilon + \frac{Ct^{-\delta}}{\varepsilon\delta}.$$

Letting $t \to +\infty$, we see that

$$\limsup_{t\to+\infty} |g_t(1) - g(1)| \leq CM\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $g_t(1) \to g(1)$ as $t \to +\infty$, as desired. □

4. Concluding remarks

We now try to connect the proof of the prime number theorem above, with the one in Chapter 7 of [4].

In Chapter 7 of [4], instead of proving directly the asymptotic (5) of $\psi(x)$, they proceeded in first establishing an asymptotic of $\psi_1(x)$, which by definition is the indefinite integral of ψ that is zero when $x = 1$. In other words,

$$\psi_1(x) := \int_1^x \psi(y)dy,$$

and (5) was deduced in [4] by first showing that

$$\psi_1(x) \sim \frac{x^2}{2}. \tag{17}$$

Now we observe that Proposition 4 can be rewritten as

$$-\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty \psi(e^t)e^{-st}dt.$$

Since $\psi(e^t)dt = e^{-t}d(\psi_1(e^t))$, we can integrate by parts in the last integral, and obtain

$$-\frac{\zeta'(s)}{s\zeta(s)} = (s+1)\int_0^\infty \psi_1(e^t)e^{-(s+1)t}dt.$$

(The boundary terms vanish.) In particular, dividing both sides by $(s + 1)$, we obtain

$$\int_0^\infty \psi_1(e^t)e^{-(s+1)t}dt = -\frac{\zeta'(s)}{\zeta(s)}\frac{1}{s(s+1)}.$$

Now this says the Laplace transform of the function $e^{-t}\psi_1(e^t)$ is $-\frac{\zeta'(s)}{\zeta(s)}\frac{1}{s(s+1)}$. But this Laplace transform can be inverted by the following Bromwich integral: if $|f(t)| \leq Ce^{at}$ for all $t \geq 0$, and $F(s)$ is the Laplace transform of $f(t)$, i.e.

$$F(s) = \int_0^\infty f(t)e^{-st}dt \quad \text{for } \operatorname{Re} s > a,$$

then whenever $c > a$, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \quad \text{for almost every } t > 0.$$

(This follows from the Fourier inversion formula, applied to the function that is $f(t)e^{-ct}$ if $t \geq 0$, and zero if $t < 0$.) We make use of this inversion formula as follows.

Recall that $\psi(x)/x$ is bounded. This implies readily that $\psi_1(x)/x^2$ is bounded. Hence $e^{-t}\psi_1(e^t)$ is bounded by Ce^t. Taking $a = 1$ in the above argument, we see that

$$e^{-t}\psi_1(e^t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{e^{st}}{s(s+1)} ds.$$

Multiplying both sides by e^t, and then rewrite $x = e^t$, we obtain a formula for $\psi_1(x)$ in terms of ζ:

$$\psi_1(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{s(s+1)} ds \tag{18}$$

and this is precisely Proposition 2.3 of Chapter 7 of [4]. So that Proposition should really be interpreted as a computation for the Laplace transform of $\psi_1(x)/x$!

Finally, instead of arguing indirectly (as we did) that (5) holds, in Chapter 7 of [4], they proceeded to obtain the asymptotics (17) of ψ_1 directly using (18). To do so, one would 'compute' (18) as one would usually do with Bromwich integral: one shifts the contour of integration to the left, across the singularities of $-\frac{\zeta'(s)}{\zeta(s)}\frac{1}{s(s+1)}$ as much as possible. It turns out one only needs to shift the contour past the first singularity at $s = 1$; the principle contribution comes from the pole $s = 1$, and everything else is a small error compared to that. To actually carry this out, one needs upper bounds for ζ'/ζ near the critical line $\operatorname{Re} s = 1$. This translates into lower bounds for ζ on the line $\operatorname{Re} s = 1$ (which is a quantitative version of the non-vanishing of ζ on $\operatorname{Re} s = 1$), and upper bounds for ζ' on the line $\operatorname{Re} s = 1$ (which was a quantitative version of the analytic continuation of ζ past the line $\operatorname{Re} s = 1$, as we have seen in Chapter 6 of [4]). With these estimates for ζ, they finish the proof of the asymptotics (17) of ψ_1, and hence the proof of the prime number theorem in Chapter 7 of [4].

References

[1] P. Erdös, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949).

[2] D. J. Newman, *Simple analytic proof of the prime number theorem*, Amer. Math. Monthly **87** (1980), no. 9, 693–696.

[3] Atle Selberg, *An elementary proof of the prime-number theorem*, Ann. of Math. (2) **50** (1949), 305–313.

[4] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003.

[5] D. Zagier, *Newman's short proof of the prime number theorem*, Amer. Math. Monthly **104** (1997), no. 8, 705–708.